DOCUMENT RESUME

ED 233 067 TM 830 515

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TITLE Analysis of Balanced 2 [to the kth power] Mirror-Image Designs.
PUB DATE 14 Apr 83
NOTE 10p.; Paper presented at the Annual Meeting of the American Educational Research Association (67th, Montreal, Quebec, April 11-15, 1983).
PUB TYPE Speeches/Conference Papers (150) -- Reports - Research/Technical (143)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Aptitude Treatment Interaction; *Data Analysis; Hypothesis Testing; Individual Characteristics; *Research Design; Research Methodology; Simulation; *Statistical Analysis
IDENTIFIERS *Mirror Image Designs; *Repeated Measures Design

ABSTRACT

A 2-to-the-kth-power mirror-image design is defined as one in which repeated observations of subjects occurs among the levels of a usual 2-to-the-kth-power design, but there is the restriction that no subject may receive a given level of any factor more than once. Such a restriction might arise, for example, if a subject's response is expected to be affected artifically if any given level of a factor is presented two or more times. This paper describes an analysis for data which arise from such designs. Only the balanced case is treated. Subjects in a 2-to-the-kth-power mirror-image design may be divided into 2-to-the-k-minus-1 groups, each of which receives a specific, unique pair of treatments. This grouping arises from the restriction in the repetition of levels admissible in the design. Each subject thus yields two scores under two combinations of distinct levels of the factors. It is assumed that the populations the groups represent are homogeneous bivariate normal. Significance tests are developed in the paper for the usual main and interaction effects associated with a 2-to-the-kth-power design. A fully worked example is included. (Author)

Analysis of Balanced 2^k Mirror-Image Designs

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1. Introduction

Consider 2^k experimental designs in which repeated observations of subjects occurs but there is the restriction that no subject may receive a given level of a factor more than once. Such a restriction might arise, for example, if subjects' responses are expected to be affected artificially if any given level of a factor is presented two or more times. The purpose of this paper is to describe an analysis for data arising from such designs, called here 2^k mirror-image designs.

As an example of such an experiment, consider a study which focuses on main and interaction effects of four characteristics of persons; age, race, sex, and citizenship; on the social distance accorded them by judges. For each of the four factors, two conditions are simulated and presented to the subjects who are to respond on a scale designed to assess social distance. It is reasonable for any given judge to rate simulations representing combinations of different levels of each of the factors, but these assessments may be affected artificially if a judge were to rate two simulations for which any factor did not vary (e.g., through a focus on a restricted set of conditions in making comparisons among the simulations). It should be noted, in the extreme, that complete repetition of all sixteen simulations would render transparent the nature of the experiment to each judge.

Paper presented at the Convention of the American Educational Research Association, Montreal, Canada, April 14, 1983.

Printed in U.S.A.

Given the restriction in the repetition of levels which are admissible in any 2^k mirror-image design, a total of 2^{k-1} groups is required to receive the various allowable pairs of treatment conditions. For example, in the 2^4 design given above, this becomes 2^3 or eight groups. If we denote the factors as A, B, C, and D, and their levels as 1 and 2, these eight groups would receive: (1) A1B1C1D1 and A2B2C2D2, (2) A1B1C1D2 and A2B2C2D1, (3) A1B1C2D1 and A2B2C1D2, (4) A1B1C2D2 and A2B2C1D1, (5) A1B2C1D1 and A2B1C2D2, (6) A1B2C1D2 and A2B1C2D1, (7) A1B2C2D1 and A2B1C1D2, and (8) A1B2C2D2 and A2B1C1D1. The nature of the repetition is diagramed below.

Table 1

Grouping Pattern for 2^4 Mirror-Image Design

Factor A	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
Factor B	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2
Factor C	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2
Factor D	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
Group 1	x															x
Group 2		x													x	
Group 3			x											x		
Group 4				x									x			
Group 5					x							x				
Group 6						x					x					
Group 7							x			x						
Group 8								x	x							

x denotes the combinations of factors received by that group.

In general, such a design will consist of 2^{k-1} groups of subjects, each group receiving two combinations of conditions. Assuming random assignment of subjects to groups, normality for the distributions of responses, and homogeneity of variances and covariances among repeated measures, significance tests can be developed for the usual main and interaction effects in a 2^k mirror-image design. The description here is restricted to balanced designs (all 2^{k-1} groups of equal size). Finally, a complete numerical example is given.

2. Variances of the Effects

Each effect, main or interaction, is based on one degree of freedom, so that a single contrast among group means can be written which corresponds to each effect. The sampling variances for these contrasts are considered separately according to the level of the effect.

For each main effect, the nature of the design ensures that the contrast coefficients (+1,-1) are opposite in sign for each group. Letting V represent the variance of any of the 2^k cells and G represent the covariance for any of the 2^{k-1} groups, the variance of any main effect contrast contains 2^k terms of the form V/n and 2^k terms of the form -G/n. Thus, the variance of a main effect contrast is $2^k/n$ (V - G).

The coefficients of the contrast for any first-order interaction may be found as the products of the respective contrast coefficients for the appropriate pair of main effect contrasts. Since at the main effect level the coefficients for any one group are opposite in sign, the products for any one group at the first order interaction level are either of the form (+1,-1)(+1,-1)=(+1,+1); (+1,-1)(-1,+1)=(-1,-1); (-1,+1)(+1,-1)=(-1,-1); or (-1,+1)(-1,+1)=(+1,+1). Thus, the coefficients for any group at the first order interaction level are like in sign. Using the same notation, then, the variance of any first order interaction contrast contains 2^k terms of the form V/n and 2^k terms of the form G/n. Thus, the variance of a first order interaction contrast is $2^k/n$ (V + G).

The coefficients for any second order interaction contrast may be found as the products of the respective contrast coefficients for an appropriate pair of contrasts, one of which is a first order interaction and the other a main effect. At the first order interaction level the coefficients are of like sign and at the main effect level the coefficients are of unlike sign. Thus the products for any one group at the second order interaction level are either of the form (+1,+1)(+1,-1)=(+1,-1); (+1,+1)(-1,+1)=(-1,+1); (-1,-1)(+1,-1)=(-1,+1); or (-1,-1)(-1,+1)=(+1,-1). Since the coefficients for any one group at the second order level are opposite in sign, the discussion for the main effect level applies here and the variance of a contrast at the second order interaction level is $2^k/n\ (V - G)$.

In general the coefficients for a contrast at any level of interaction may be found as the products of the respective contrast coefficients for an appropriate pair of contrasts, one at the next lower level of interaction and the other a main effect. Therefore, at any odd interaction level the variance of a contrast is $2^k/n\ (V + G)$ and at any even interaction level as well as for main effects the variance of a contrast is $2^k/n\ (V - G)$.

3. Estimates of the Variance and Covariance Functions

The variances of the contrasts of a 2^k mirror-image design have been found to be of the form $2^k/n\ (V + G)$ or $2^k/n\ (V - G)$. In this section estimators of each of these functions of variance and covariance are discussed.

For any one group let S_i represent the sum of the two observations for the ith subject and D_i represent the difference, so that

$$S_i = X_{1i} + X_{2i} \qquad \text{and} \qquad D_i = X_{1i} - X_{2i}$$

The expectation of the mean square for S is:

$$E\,[\,\mathrm{SUM}\,(S_i - \bar{S})^2\,]/(n-1)$$

$$= E\,[\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1 + X_{2i} - \bar{X}_2)^2\,]/(n-1)$$

$$= E\,[\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1)^2 + \mathrm{SUM}\,(X_{2i} - \bar{X}_2)^2 + 2\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)\,]/(n-1)$$

$$= 2V + 2G.$$

Pooled over the 2^{k-1} groups, then, the error mean square for S (MS_S) estimates 2(V+G) with $2^{k-1}(n-1)$ degrees of freedom.

The expectation of the mean square for D for a given group is

$$E\,[\,\mathrm{SUM}\,(D_i - \bar{D})^2\,]/(n-1)$$

$$= E\,[\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1 - X_{2i} + \bar{X}_2)^2\,]/(n-1)$$

$$= E\,[\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1)^2 + \mathrm{SUM}\,(X_{2i} - \bar{X}_2)^2 - 2\,\mathrm{SUM}\,(X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)\,]/(n-1)$$

$$= 2V - 2G.$$

Pooled over the 2^{k-1} groups, then, the error mean square for D (MS_D) estimates 2(V-G) with $2^{k-1}(n-1)$ degrees of freedom.

4. Significance Tests

Each of the contrasts discussed in section 2 is normally distributed under the assumptions given in section 1 and section 3 provides a means of arriving at unbiased estimates of their variances. For main effect and even order interaction contrasts the variance is $2^k/n$ (V-G) which is estimated by

$(2^{k-1}/n)\,MS_D$; for odd order interaction contrasts the variance is $2^k/n\,(V+G)$ which is estimated by $(2^{k-1}/n)\,MS_S$.

Thus t ratios with $2^{k-1}(n-1)$ degrees of freedom may be formed to provide significance tests. If C is a main effect or an even order interaction contrast then $t = C(est)/[MS_D(2^{k-1}/n)]^{1/2}$ and if C is an odd order interaction contrast then $t = C(est)/[MS_S(2^{k-1}/n)]^{1/2}$.

5. Example

In this section a numerical example is given using data collected following a 2^3 mirror-image design plan. In this study (the authors are grateful to L. Brachfeld and H. Teglasi-Golubcow for permission to use their data) main and interaction effects for three factors on admissions ratings for applicants to educational programs were studied. The factors were (1) aptitude of applicant, (2) quality of letters of recommendation, and (3) whether or not the applicant waived access to the letters. Each factor was simulated at two levels and thus eight different packets of credentials were used. Four groups of judges were formed randomly. Each judge rated two "applicants" on likelihood of a positive admission decision. The table below gives the packets presented to each group of judges.

Group	Packet	Aptitude	Letter	Waiver
1	1	High	Superior	Absent
	2	Average	Average	Present
2	3	High	Superior	Present
	4	Average	Average	Absent
3	5	High	Average	Absent
	6	Average	Superior	Present
4	7	High	Average	Present
	8	Average	Superior	Absent

Judges were solicited by mail. The groups were formed randomly and were of equal size. Likelihood of a positive decision was scaled from one to nine with nine being most likely.

The data are:

Group 1 (packet1, packet2): (8,4) (8,7) (8,7) (7,7) (9,6) (8,6) (8,3) (9,7) (8,7) (9,5) (9,5) (8,6) (9,5) (9,4) (9,7) (8,7) (8,6) (9,7) (6,5)

Means	(8.26,5.84)
Sum of Squares for Sum:	39.79
Sum of Squares for Difference:	40.63

Group 2 (packet3, packet4): (7,6) (7,4) (7,5) (7,3) (9,7) (7,3) (9,7) (9,6) (9,6) (9,7) (7,3) (8,2) (7,3) (8,6) (8,5) (7,3) (8,4) (7,3) (7,4)

Means	(7.74,4.58)
Sum of Squares for Sum:	100.11
Sum of Squares for Difference:	24.53

Group 3 (packet5, packet6): (9,9) (3,3) (9,1) (7,6) (7,7) (9,7) (6,8) (9,6) (8,7) (5,5) (9,9) (7,4) (5,7) (8,8) (5,6) (7,6) (9,8) (7,6) (6,8)

Means	(7.11,6.37)
Sum of Squares for Sum:	166.74
Sum of Squares for Difference:	93.68

Group 4 (packet7, packet8): (6,6) (9,7) (9,6) (9,8) (6,4) (4,5) (7,7) (8,7) (9,5) (7,4) (8,3) (8,3) (7,5) (9,7) (7,6) (7,7) (8,7) (8,4) (7,7)

Means	(7.53,5.68)
Sum of Squares for Sum:	89.16
Sum of Squares for Difference:	56.53

The means arrayed by cells are:

		Aptitude: High		Aptitude: Average	
Letter		Superior	Average	Superior	Average
Waiver	Absent	8.26	7.11	5.68	4.58
	Present	7.74	7.53	6.37	5.84

The sum of the sums of squares for Sum is 395.79 which with df = 4(18) = 72 yields a mean square for Sum of 5.50; the sum of the sums of squares for Difference is 215.37 which yields a mean square for Difference of 2.99. With unit coefficients the contrasts, denominators, and t ratios are:

Source	Contrast	Denominator	t
Aptitude	8.16	.79	10.28
Letter	3.00	.79	3.78
Waiver	- 1.84	.79	- 2.32
Aptitude x Letter	- .26	1.08	- .24
Aptitude x Waiver	2.05	1.08	1.91
Letter x Waiver	1.53	1.08	1.42
Aptitude x Letter x Waiver	.37	.79	.46

Only the three main effect contrasts are significantly different from zero.

TM 830 515

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